On the critical communication range under node placement with vanishing densities

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Abstract— **We consider the random network where n points are placed independently on the unit interval $[0,1]$ according to some probability distribution function F. Two nodes communicate with each other if their distance is less than some transmission range. When F admits a continuous density f with $f_\star = \inf\left(f(x),\ x \in [0,1]\right) > 0$, the property of graph connectivity for the underlying random graph is known to admit a strong critical threshold. Through a counterexample, we show that only a weak critical threshold exists when $f_\star = 0$ and we identify it. Implications for the critical transmission range are discussed.**

Keywords: Geometric random graphs, Non-uniform node placement, Vanishing density, Weak critical thresholds, Zero-one laws, Critical transmission range.

I. Introduction

The following one-dimensional random network model has been discussed in a number of contexts, e.g., see [2, 3, 4, 5, 6, 9, 10, 13, 14, 17, 18, 19] (and references therein): The network comprises n (communication) nodes which are placed independently on the interval $[0,1]$ according to some probability distribution F. Two nodes are said to communicate with each other if their distance is less than some transmission range $\tau > 0$.

A basic question concerns the existence of a typical behavior for the property of graph connectivity as n becomes large and the transmission range τ is scaled appropriately with n. This is achieved by means of scalings or range functions $\tau : \mathbb{N}_0 \to \mathbb{R}_+ : n \to \tau_n$, and often results in *zero-one* laws according to which the graph is connected (resp. not connected) with a very high probability (as n becomes large) depending on how the scaling deviates from a *critical* scaling $\tau^\star$. Such critical thresholds are likely to be distribution dependent and serve as rough indicators of the smallest (so-called critical) transmission range needed to ensure network connectivity [17, 19].

The references above deal overwhelmingly with the situation when F is the *uniform* distribution on the interval $[0,1]$. In this setting it is well known [1, 2, 3, 6, 9, 10, 13, 14] that the property of graph connectivity admits a zero-one law with a *strong* (critical) threshold; more on that in Section II. Recently, the authors [11] have obtained similar results when the probability distribution F has a continuous and *non-vanishing* density f: With

$$f_\star = \inf\left(f(x),\ x \in [0,1]\right) > 0, \qquad (1)$$

we have shown that

$$\tau^\star_{F,n} = \frac{1}{f_\star} \cdot \frac{\log n}{n}, \quad n = 1,2,\ldots \qquad (2)$$

is a strong threshold for graph connectivity.

A natural question arises as to the validity and form of these results when the density f vanishes on the interval $[0,1]$ – Such situations do occur in applications, e.g., highway networks under random waypoint mobility [4, 19]. In this paper we show through simple examples that when (1) fails, the property of graph connectivity may still exhibit a zero-one law. However, the coresponding threshold is now only a *weak* critical threshold (in a technical sense to be made precise in Section II). This (weak) critical threshold is now of a much larger order than the one given at (2). Implications for resource dimensioning (via the critical transmission range) and for the non-existence of sharp phase transitions in these models are briefly discussed in Section III. The examples used here were selected for their ease of analysis. However, they are representative of many situations when f vanishes at isolated points, e.g., the stationary node distribution under the random waypoint mobility model without pause [19].

The paper is organized as follows: Section II presents the model assumptions, and the notions of strong and weak critical thresholds. Section III discusses the technical contributions of the paper. In Section IV we translate the existence of a zero-one law into asymptotic properties of maximal spacings induced by i.i.d. variates drawn from F. We continue in Section V with a useful representation of the spacings associated with the uniform distribution. This representation, which is given in terms of i.i.d. exponentially distributed rvs, is key to establishing the results in Section VI.

II. Model assumptions and definitions

All the rvs under consideration are defined on the *same* probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{X_i,\ i = 1,2,\ldots\}$ denote a sequence of i.i.d. rvs which are distributed on the unit interval $[0,1]$ according to some common probability distribution function F. We assume that F admits a density function $f:[0,1]\to\mathbb{R}_+$ which is *continuous* on the interval $[0,1]$.

For each $n=2,3,\ldots$, we think of $X_1,\ldots,X_n$ as the locations of the n nodes, labelled $1,\ldots,n$, in the interval $[0,1]$. Given a fixed transmission range $\tau>0$, two nodes are said to communicate if their distance is at most τ. In other words, we can think of nodes i and j as connected if $|X_i-X_j|\le\tau$, in which case an undirected edge is said to exist between them. This notion of connectivity gives rise to the undirected geometric random graph $\mathbb{G}(n;\tau)$. The geometric random graph $\mathbb{G}(n;\tau)$ is said to be *(path) connected* if every pair of distinct nodes can be linked by at least one path over the edges of the graph, and we write

$$P(n;\tau):=\mathbb{P}\left[\mathbb{G}(n;\tau)\text{ is connected}\right].$$

Obviously $P(n;\tau)=1$ whenever $\tau\ge1$.

Some terminology is needed before we can start the discussion: A range function τ is defined as any mapping $\tau:\mathbb{N}_0\to\mathbb{R}_+$. A range function $\tau^\star$ is said to be a *weak* (critical) threshold (for the property of graph connectivity) [13, p. 376] if

$$\lim_{n\to\infty}P(n;\tau_n)=\begin{cases}0 & \text{if } \lim_{n\to\infty}\frac{\tau_n}{\tau_n^\star}=0\\[2ex] 1 & \text{if } \lim_{n\to\infty}\frac{\tau_n}{\tau_n^\star}=\infty\end{cases}\quad(3)$$

with range function $\tau:\mathbb{N}_0\to\mathbb{R}_+$. A much stronger conclusion than (3) is often possible, and is captured through the following definition: The range function $\tau^\star$ is said to be a *strong* (critical) threshold (for the property of graph connectivity) [13, p. 376] if for range functions $\tau:\mathbb{N}_0\to\mathbb{R}_+$ such that $\tau_n\sim c\tau_n^\star$ for some $c>0$, we have

$$\lim_{n\to\infty}P(n;\tau_n)=\begin{cases}0 & \text{if } 0<c<1\\[2ex] 1 & \text{if } 1<c.\end{cases}\quad(4)$$

It is customary to refer to the existence of range functions $\tau^\star$ satisfying (3) and (4), respectively, as weak and strong zero-one laws, respectively. This terminology reflects the fact that under (3) the one law (resp. zero law) occurs when using range functions $\tau:\mathbb{N}_0\to\mathbb{R}_+$ which are *at least* an order of magnitude larger (resp. smaller) than $\tau^\star$. On the other hand, under (4), for n sufficiently large, a communication range τ_n suitably larger (resp. smaller) than $\tau_n^\star$ ensures $P(n;\tau_n)\simeq1$ (resp. $P(n;\tau_n)\simeq0$) provided $\tau_n\sim c\tau_n^\star$ with $c>1$ (resp. $0<c<1$). This is in sharp contrast with (3) in that the one law (resp. zero law) still emerges with range functions $\tau:\mathbb{N}_0\to\mathbb{R}_+$ which are asymptotically larger (resp. smaller) than $\tau^\star$ but of the *same* order of magnitude as $\tau^\star$! It should be clear that any range function $\tau^\star$ which satisfies (4) necessarily satisfies (3).

III. The results

We set the stage for the discussion by recalling a result recently obtained by the authors in [11]; see [15] for a multi-dimensional version of this result.

Theorem 3.1: *Under the enforced assumptions with the positivity condition (1), the range function $\tau_F^\star:\mathbb{N}_0\to\mathbb{R}_+$ given by (2) is a strong threshold for the property of graph connectivity.*

When F is the uniform distribution U, we have $f_\star=1$ and we recover the well-known result that $\tau_{U,n}^\star=\frac{\log n}{n}$ is a strong critical threshold for graph connectivity under uniform node placement [1, 14].

When $f_\star=0$, a blind application of Theorem 3.1 yields $\tau_{F,n}^\star=\infty$ for all $n=1,2,\ldots$. This begs the question as to what is the appropriate analog of Theorem 3.1 when the density f vanishes. We explore this issue through the following simple example: With $p>0$, consider the probability distribution F_p given by

$$F_p(x)=x^{p+1},\quad x\in[0,1]\quad(5)$$

with corresponding density function f_p given by

$$f_p(x)=(p+1)x^p,\quad x\in[0,1].\quad(6)$$

Theorem 3.1 needs to be replaced by the following result.

Theorem 3.2: *Under (5), the property of graph connectivity admits only weak critical threshold functions, and the range function $\tau_p^\star:\mathbb{N}_0\to\mathbb{R}_+$ given by*

$$\tau_{p,n}^\star=n^{-\frac{1}{p+1}},\quad n=1,2,\ldots\quad(7)$$

is such a weak threshold function.

The random graph $\mathbb{G}(n;\tau)$ under (5) provides yet another situation where a strong critical threshold does not exist for a monotone graph property [13, Thm. 5.1, p. 382]. The remainder of the paper is devoted to establishing Theorem 3.2.

It is easy to check from Theorem 3.1 that the threshold function $n\to\frac{\log n}{n}$ is a weak threshold function, a *robust*, albeit weak, conclusion which holds across *all* distributions F satsifying (1). However, with F given by (5), the critical threshold given by (7) is now of a much larger order since

$$\frac{\log n}{n}=o\left(n^{-\frac{1}{p+1}}\right).$$

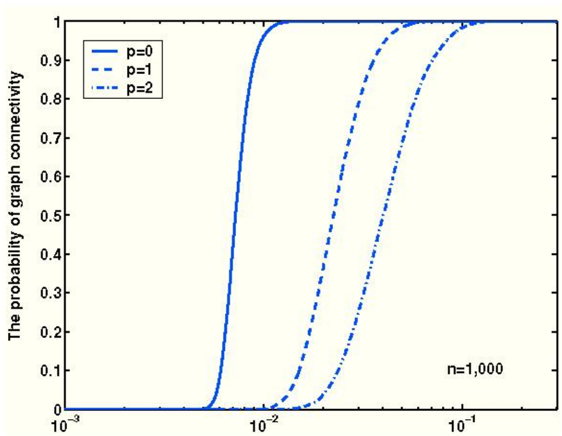
Implications for resource dimensioning in two-dimensional ad-hoc networks were already discussed in the references [17, 19], and take here the following form: As will become apparent from the comments following Lemma 4.2, critical thresholds serve as proxy for the critical transmission range when n is large. Thus, under a node placement with a vanishing density such as (5), we see that the critical transmission range is orders of magnitude *larger* than would otherwise have been the case when (1) holds, resulting in *higher* minimum power levels to ensure connectivity. Similar *qualitative* conclusions were already pointed out by Santi [19, Thm. 4] for two-dimensional networks under the random waypoint mobility model without pause. In one dimension, the corresponding stationary spatial node density is given by

$$f_{\mathrm{RWP}}(x)=6\ x(1-x),\quad 0\le x\le1.\quad(8)$$

Here, under (5) we can go beyond qualitative statements and give *precise* information on the *order* of the asymptotics for the critical transmission range.

Although the distribution (5) was selected because its simpler form facilitated the analysis, it is nevertheless representative of vanishing densities such as (8). Indeed, both Theorems 3.1 and 3.2 derive from limiting properties of the maximal spacing under F. Such properties are influenced by the behavior of the density in the vicinity of its minimum point [12, p. 519.]: The densities (6) (with $p = 1$) and (8) have similar behavior near $x = 0$ since $f_{\mathrm{RWP}}(x) \sim 6x$ as $x \simeq 0$. Thus, the results obtained here suggest that this model requires a much larger critical transmission range function given by

$$\tau^\star_{\mathrm{RWP},n} = \frac{1}{\sqrt{n}}, \quad n = 1, 2, \ldots.$$



Under uniform node placement, the number of breakpoint users[1] is known to converge to a Poisson rv under the appropriate critical scaling [7, 10]. This property crisply captures the fact that the phase transition usually associated with strong zero-one laws is a very sharp one indeed [7, 8, 10]. However, the absence of strong critical thresholds under (5) precludes such Poisson convergence, and essentially rules out the possibility that the corresponding phase transition will be sharp in this case.

These conclusions are already apparent from the limited simulation results presented above where nodes are placed according to F_p with $p = 0, 1, 2$; the case $p = 0$ corresponds to the uniform distribution. For each $p = 0, 1, 2$, the figure displays the corresponding plot of $P(n, \tau)$ as a function of τ (in base 10 log-scale) for $n = 1,000$. In each case we generated $K = 10,000$ mutually independent configurations, each configuration consisting of n points on the interval $[0, 1]$ drawn independently according to F_p. We compute the value $P(n, \tau)$ as the ratio $X_K(n, \tau)/K$ where $X_K(n, \tau)$ records the number of configurations among these K configurations which result in a connected graph when the transmission range is τ. As expected, the phase transition is much sharper for $p = 0$ than for positive p. These displays also suggest that the sharpness of the phase transition decreases with increasing p. However, at the time of this writing, we are not in a position to offer precise quantitative results validating this claim.

[1]For each $k = 1, 2, \ldots, n$, node k is a breakpoint user in $\mathbb{G}(n; \tau)$ if (i) it is not the leftmost node in $[0, 1]$ and (ii) there is no other node in the random interval $[X_k - \tau, X_k]$.

IV. Preliminaries

Fix $n = 2, 3, \ldots$ and τ in $(0, 1)$. With the node locations $X_1, \ldots, X_n$, we associate the rvs $X_{n,1}, \ldots, X_{n,n}$ which are the locations of the n users arranged in increasing order, i.e., $X_{n,1} \le \ldots \le X_{n,n}$ with the convention $X_{n,0} = 0$ and $X_{n,n+1} = 1$. The rvs $X_{n,1}, \ldots, X_{n,n}$ are the *order statistics* associated with the n i.i.d. rvs $X_1, \ldots, X_n$. We also define the spacings

$$L_{n,k} := X_{n,k} - X_{n,k-1}, \quad k = 1, \ldots, n+1. \tag{9}$$

Interest in these spacings derives from the observation that the graph $\mathbb{G}(n; \tau)$ is connected if and only if $L_{n,k} \le \tau$ for all $k = 2, \ldots, n$, so that

$$P(n; \tau) = \mathbb{P}\left[M_n \le \tau\right] \tag{10}$$

where

$$M_n := \max\left(L_{n,k},\ k = 2, \ldots, n\right). \tag{11}$$

Let the range function $\tau^\star : \mathbb{N}_0 \to \mathbb{R}_+$ be considered as a candidate threshold function for graph connectivity. Then, for any other range function $\tau : \mathbb{N}_0 \to \mathbb{R}_+$, we have

$$P(n; \tau_n) = \mathbb{P}\left[\frac{M_n}{\tau^\star_n} \le \frac{\tau_n}{\tau^\star_n}\right], \quad n = 1, 2, \ldots \tag{12}$$

Simple criteria are now given for checking whether the range function $\tau^\star$ is indeed a weak or a strong threshold. We do so under the natural assumption that there exists an $\mathbb{R}_+$-valued rv L such that

$$\frac{M_n}{\tau^\star_n} \Longrightarrow_n L \tag{13}$$

where $\Longrightarrow_n$ denotes convergence in distribution with n going to infinity.

Lemma 4.1: *If (13) holds with $\mathbb{P}[L = 0] = 0$, then the range function $\tau^\star : \mathbb{N}_0 \to \mathbb{R}_+$ is a weak threshold.*

The next result characterizes strong thresholds in terms of asymptotic properties of the maximal spacings (11).

Lemma 4.2: *The range function $\tau^\star : \mathbb{N}_0 \to \mathbb{R}_+$ is a strong threshold if and only*

$$\frac{M_n}{\tau^\star_n} \xrightarrow{P}_n 1 \tag{14}$$

where $\xrightarrow{P}_n$ denotes convergence in probability with n going to infinity.

Lemmas 4.1 and 4.2 have easy proofs; they are omitted in the interest of brevity with details available in [7].

For each $n = 2, 3, \ldots$, the *critical transmission range* for the n node network is defined as the rv R_n given by

$$R_n := \min\left(\tau > 0 : \mathbb{G}(n; \tau) \text{ is connected}\right).$$

In short, R_n is the smallest transmission range that ensures that the node set $X_1, \ldots, X_n$ forms a connected network. The obvious identity

$$R_n = M_n$$

leads to the following operational interpretation of critical thresholds: By Lemma 4.2, the range function $\tau^\star : \mathbb{N}_0 \to \mathbb{R}_+$ is a strong critical threshold if and only if $R_n \sim \tau_n^\star$ for n large in some appropriate distributional sense (formalized at (14)). On the other hand, if $\tau^\star$ is a weak critical threshold, then Lemma 4.1 only states that $R_n \sim \tau_n^\star L$ for n large with a non-zero (possibly non-degenerate) rv L. In either case, but with different degrees of accuracy, the critical threshold serves as a proxy or estimate of the critical transmission range for the many node networks.

V. Representing the maximal spacing

In addition to the i.i.d. $[0,1]$-valued rvs $\{X_i,\ i=1,2,\ldots\}$ distributed according to F_p, consider a second collection of i.i.d., rvs $\{U_i,\ i=1,2,\ldots\}$ which are all uniformly distributed on $[0,1]$. In analogy with the notation introduced above, for each $n=2,3,\ldots$, we introduce the order statistics $U_{n,1},\ldots,U_{n,n}$ associated with the n i.i.d. rvs $U_1,\ldots,U_n$ and we again use the convention $U_{n,0}=0$ and $U_{n,n+1}=1$. It is well known that

$$(X_1,\ldots,X_n) =_{st} (F_p^{-1}(U_1),\ldots,F_p^{-1}(U_n)) \qquad (15)$$

where $F_p^{-1} : [0,1] \to [0,1]$ is the inverse mapping of F given by

$$F_p^{-1}(x) = x^{\frac{1}{p+1}}, \quad x \in [0,1].$$

Therefore, it is easy to see that

$$(X_{n,1},\ldots,X_{n,n}) =_{st} (F_p^{-1}(U_{n,1}),\ldots,F_p^{-1}(U_{n,n})).$$

It is now plain that

$$\begin{aligned} & (L_{n,k},\ k=2,\ldots,n) \\ =_{st} \ & (F_p^{-1}(U_{n,k}) - F_p^{-1}(U_{n,k-1}),\ k=2,\ldots,n) \\ = \ & \left((U_{n,k})^{\frac{1}{p+1}} - (U_{n,k-1})^{\frac{1}{p+1}},\ k=2,\ldots,n \right). \end{aligned}$$

In order to take advantage of this last equivalence, we introduce a collection of $\{\xi_j,\ j=1,2,\ldots\}$ of i.i.d. rvs which are exponentially distributed with unit parameter, and set

$$T_0 = 0,\ T_k = \xi_1 + \ldots + \xi_k, \quad k=1,2,\ldots.$$

For all $n=1,2,\ldots$, the stochastic equivalence

$$(U_{n,1},\ldots,U_{n,n}) =_{st} \left(\frac{T_1}{T_{n+1}},\ldots,\frac{T_n}{T_{n+1}} \right) \qquad (16)$$

is known to hold [16, p. 403] (and references therein). Therefore, upon defining

$$V_k := (T_k)^{\frac{1}{p+1}} - (T_{k-1})^{\frac{1}{p+1}}, \quad k=1,2,\ldots,$$

we get

$$\begin{aligned} & (F_p^{-1}(U_{n,k}) - F_p^{-1}(U_{n,k-1}),\ k=2,\ldots,n) \\ =_{st} \ & \left(\frac{V_k}{(T_{n+1})^{\frac{1}{p+1}}},\ k=2,\ldots,n \right). \end{aligned}$$

Consequently, the distributional equivalence

$$M_n =_{st} \frac{M_n^\star}{(T_{n+1})^{\frac{1}{p+1}}} \qquad (17)$$

holds where we have defined

$$M_n^\star := \max(V_k,\ k=2,\ldots,n). \qquad (18)$$

VI. A proof of Theorem 3.2

The range function $\tau_p^\star : \mathbb{N}_0 \to \mathbb{R}_+$ is the one given by (7). We start with the following key representation that flows from (17)–(18), namely

$$\frac{M_n}{\tau_{p,n}^\star} =_{st} \left(\frac{n}{T_{n+1}} \right)^{\frac{1}{p+1}} \cdot M_n^\star, \quad n=1,2\ldots. \qquad (19)$$

The proof proceeds according to three distinct steps.

A. The range function $\tau_p^\star$ is a weak threshold

By the Strong Law of Large Numbers, we already have

$$\lim_{n\to\infty} \frac{T_{n+1}}{n} = 1 \quad a.s. \qquad (20)$$

Moreover, the sequence $\{M_n^\star,\ n=2,3,\ldots\}$ being monotone, we have the a.s. convergence

$$\lim_{n\to\infty} M_n^\star = \sup(V_k,\ k=2,\ldots) =: M^\star \quad a.s. \qquad (21)$$

If we could show that $M^\star$ is a.s. finite with $M^\star > 0$ a.s., then we readily see from (19), (20) and (21) that

$$\frac{M_n}{\tau_n^\star} \Longrightarrow_n M^\star. \qquad (22)$$

Therefore, (13) holds with $\tau^\star = \tau_p^\star$ and $L =_{st} M^\star$, and by Lemma 4.1, the range function $\tau_p^\star : \mathbb{N}_0 \to \mathbb{R}_+$ is a weak threshold.

We now show that $M^\star$ is a.s. finite with $M^\star > 0$ a.s.: First, we note that $M^\star \geq V_2$. But $V_2 = 0$ if and only if $T_2 = T_1$, which occurs if and only if $\xi_2 = 0$, this last event occuring with zero probability. Consequently $V_2 > 0$ a.s. and $M^\star > 0$ a.s., as needed.

Next, fix $k=2,3,\ldots$ and for notational convenience, set $q = \frac{p}{p+1}$ and $r = \frac{p+1}{p} = q^{-1}$. It is plain that

$$\begin{aligned} V_k &= (T_k)^{\frac{1}{p+1}} - (T_{k-1})^{\frac{1}{p+1}} \\ &= \frac{1}{p+1} \int_{T_{k-1}}^{T_k} t^{-q} dt \\ &\leq \frac{1}{p+1} \int_{T_{k-1}}^{T_k} (T_{k-1})^{-q}\, dt \\ &= \frac{1}{p+1} \cdot (T_{k-1})^{-q} \cdot \xi_k \end{aligned} \qquad (23)$$

with

$$(T_{k-1})^{-q} \cdot \xi_k = \left(\frac{k}{T_{k-1}} \cdot \frac{\xi_k^r}{k} \right)^q.$$

The Strong Law of Large Numbers immediately implies

$$\lim_{k\to\infty} \frac{k}{T_{k-1}} = 1 \quad a.s.$$

as pointed out earlier. Applying again the Strong Law of Large Numbers, this time to the sequence of i.i.d. rvs $\{\xi_k^r,\ k = 1, 2, \ldots\}$, we find

$$\lim_{k\to\infty} \frac{1}{k}\sum_{\ell=1}^{k} \xi_\ell^r = \mathbb{E}\left[\xi_1^r\right] \quad a.s.$$

The exponential distribution having finite moments of all orders, we obviously have $\mathbb{E}\left[\xi_1^r\right]$ finite, whence

$$\lim_{k\to\infty} \frac{\xi_k^r}{k} = 0 \quad a.s.$$

according to a standard argument.

With the help of these observations, we conclude that

$$\lim_{k\to\infty} \left(T_{k-1}\right)^{-q} \cdot \xi_k = 0 \quad a.s.$$

whence $\lim_{k\to\infty} V_k = 0$ a.s. by virtue of (23). Therefore, there exists a (sample dependent) positive integer ν which is a.s. *finite* such that $M^\star = V_\nu$ and $M^\star$ is a.s. finite.

B. The range function $\tau_p^\star$ is not a strong threshold

Pick ε in $(0, 1)$ and $n = 2, 3, \ldots$. Obviously, $M_n^\star \geq V_2$, so that

$$\mathbb{P}\left[M_n^\star > 1+\varepsilon\right] \geq \mathbb{P}\left[V_2 > 1+\varepsilon\right] > 0$$

and $M^\star > 1$ with positive probability! Thus, (14) fails and by Lemma 4.2 the range function $\tau_p^\star : \mathbb{N}_0 \to \mathbb{R}_+$ is *not* a strong threshold for the property of graph connectivity in $\mathbb{G}(n; \tau)$.

C. There exists no strong threshold

The argument proceeds by contradiction: Assume that a strong threshold function does exist, say $\sigma : \mathbb{N}_0 \to \mathbb{R}_+$, in which case we have $\frac{M_n}{\sigma_n} \stackrel{P}{\to}_n 1$ by Lemma 4.2. Using (22), we readily conclude

$$\frac{\sigma_n}{\tau_{p,n}^\star} \Longrightarrow_n M^\star$$

as we note

$$\frac{\sigma_n}{\tau_{p,n}^\star} = \frac{\sigma_n}{M_n} \cdot \frac{M_n}{\tau_{p,n}^\star}, \quad n = 2, 3, \ldots.$$

The limit $\lim_{n\to\infty} \frac{\sigma_n}{\tau_{p,n}^\star}$ being *deterministic*, we have a contradiction since $M^\star$ is not a degenerate rv. Consequently, there cannot be strong threshold functions for the property of graph connectivity.

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